Combinatorial Sets of Reals, I

Creatures, Templates and Coherent Systems

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Section Set Theory & Topology

Classical real analysis & combinatorial sets of reals:

The works of

- Du Bois-Reymond, Riemann, Cantor,
- Hausdorff, Hilbert, Rothberger,
- Sierpinski, Souslin, Borel,
- many others, as well as
- many of our contemporaries

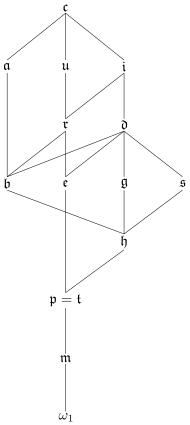
leading to interesting combinatorial structures on $\mathbb{N}^{\mathbb{N}}$ and $[\mathbb{N}]^{\infty}$.

Elementary(?!) combinatorics of $[\mathbb{N}]^\infty$ and $\mathbb{N}^\mathbb{N}$:

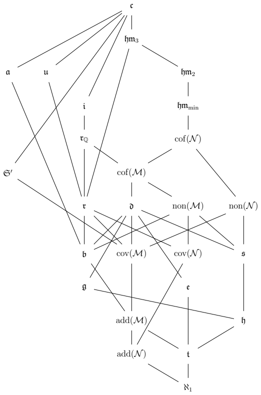
Combinatorics of eventual dominance, $\leq^*$, and basic set theoretic operations on the above spaces:

- unbounded families
- dominating families
- almost disjointness
- independence
- towers
- refining systems of a.d. families
- splitting families
- ideal independent families
- $\dots$

Some classical characteristics



More classical characteristics ... and thanks to Thilo Weinert for the diagram!



Independence

The infinitary combinatorics captured by the above sets of reals is complex and certainly testing the boundaries of ZFC, as witnessed by the abundance of independence among these structures, and the associated combinatorial cardinal characteristics.

A richness of combinatorial structures and four directions of research

- ZFC dependencies and independence; Constellations;
- Spectra and Strong Maximality
- Projective complexity
- Combinatorics at the uncountable and the so called higher Baire spaces

$\mathfrak{i}$ vs. $\mathfrak{u}$

In the Miller model $\mathfrak{u} < \mathfrak{i}$, while Shelah devised a special ${}^\omega\omega$-bounding poset the countable support iteration of which produces a model of $\mathfrak{i} = \aleph_1 < \mathfrak{u} = \aleph_2$ (and an instance of strong maximality).

$\mathfrak{a}$ vs. $\mathfrak{u}$

In the Cohen model $\mathfrak{a} < \mathfrak{u}$, while assuming the existence of a measurable one can show the consistency of $\mathfrak{u} < \mathfrak{a}$. The use of a measurable has been eliminated by Guzman and Kalajdzievski.

$\mathfrak{a}$ vs $\mathfrak{i}$

In the Cohen model $\mathfrak{a} < \mathfrak{i} = \mathfrak{c}$.

Question:

Is it consistent that $\mathfrak{i} < \mathfrak{a}$?

Theorem (J. Cruz-Chapital, V.F., O. Guzman, J, Supina, 2022)

It is relatively consistent, that $\mathfrak{i} < \mathfrak{a}_T$.

The bounding and the splitting numbers

- Balcar, Pelant, Simon established the consistency of $\mathfrak{s} < \mathfrak{b}$.
- In the lack of a ZFC proof of $\mathfrak{s} \leq \mathfrak{b}$, S. Shelah introduced the powerful technique of creature forcing, establishing the consistency of
$$\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2.$$

Start over a model of CH. Proceed with a countable support iteration of proper forcing notions $\langle \mathbb{P}_\alpha : \alpha \leq \omega_2 \rangle$. For each $\alpha \leq \omega_2$ let $V_\alpha = V^{\mathbb{P}_\alpha}$ and let $V = V_0$ be the ground model then:

1. $\forall \alpha < \omega_2 \exists r_{\alpha+1} \in V_{\alpha+1} \cap [\omega]^\omega$ such that

$$r_{\alpha+1} \text{ is not split by } V_\alpha \cap [\omega]^\omega \text{ and}$$

2. $V_0 \cap {}^\omega\omega$ remains unbounded in $V_{\omega_2} \cap {}^\omega\omega$.

Then no family of cardinality $\aleph_1$ is splitting, as for any such family $\mathscr{A}$ there is $\alpha < \omega_2$ such that $\mathscr{A} \subseteq V_\alpha \cap [\omega]^\omega$ and so $r_{\alpha+1}$ is not split by $\mathscr{A}$. Thus, $\mathfrak{s} = \aleph_2$, while by property (1) we have $\mathfrak{b} = \aleph_1$ in the final model.

First attempt!

Mathias forcing:

1. $(s, E) \in [\omega]^{<\omega} \times [\omega]^{\omega}$, $\max s < \min E$
2. $(s_1, E_1) \leq (s_2, E_2)$ iff s_1 end-extends s_2, $s_1 \setminus s_2 \subseteq E_2$, $E_1 \subseteq E_2$.

If G is $\mathbb{M}$ generic over V then

- $r_G = \bigcup\{s : \exists A (s, A) \in G\}$ is un-split by $V \cap [\omega]^{\omega}$,
- however its enumerating function dominates $V \cap {}^{\omega}\omega$.

Definition (Finite logarithmic measures, Shelah 1984)

Let $x \in [\omega]^{<\omega}$. A function

$$h : \mathscr{P}(x) \to \omega$$

is said to be a finite logarithmic measure on x if

$$\text{whenever } x = x_0 \cup x_1 \text{ then } h(x_0) \leq h(x) - 1 \text{ or } h(x_1) \leq h(x) - 1,$$

unless $h(x) = 0$. The value $h(x)$ is called the level of the measure.

Definition

Let $\mathbb{Q}$ be the set of all pairs (u, T) where $u \in [\omega]^{<\omega}$,

$$T = \langle (s_i, h_i) : i \in \omega \rangle$$

is a sequence of finite logarithmic measures, such that

1. $\max u < \min s_0$,
2. $\max s_i < \min s_{i+1}$,
3. $h_i(s_i) < h_{i+1}(s_{i+1})$.

Let

$$\mathrm{int}(T) = \bigcup \{ s_i : i \in \omega \}.$$

Remark

Note, if $(u, T) \in \mathbb{Q}$, then $(u, \mathrm{int}(T)) \in \mathbb{M}$.

Definition (continued)

$(u_2, T_2) \leq (u_1, T_1)$ where $T_k = \langle t_i^k : i \in \omega\rangle$ for $k = 1, 2$, $t_i^k = (s_i^k, h_i^k)$ if

1. u_2 end-extends u_1 and $u_2 \setminus u_1 \subseteq \mathrm{int}(T_1)$
2. $\mathrm{int}(T_2) \leq \mathrm{int}(T_1)$ and there is a sequence $\langle B_i : i \in \omega\rangle \subseteq [\omega]^{<\omega}$ s. t.
   - $\max u_2 < \min s_j^1$ for $j = \min B_0$, $\max B_i < \min B_{i+1}$, $s_i^2 \subseteq \bigcup\{s_j^1 : j \in B_i\}$.
   - for every $e \subseteq s_i^2$ such that $h_i^2(e) > 0$

     there is $j \in B_i$ such that $h_j^1(e \cap s_j^1) > 0$.

Properties

1. $\mathbb{Q}$ is Axiom *A*.
2. Let $A \in [\omega]^\omega$. Then
$$D_A = \{(u, T) \in \mathbb{Q} : \text{int}(T) \subseteq A \text{ or } \text{int}(T) \subseteq A^c\}$$
is dense and so
$$u_G = \bigcup \{u : \exists T (u, T) \in G\}$$
is either contained in A or in A^c. Thus, u_G is not split by $V \cap [\omega]^\omega$.
3. The countable support iteration of $\mathbb{Q}$ preserves the ground model reals unbounded. In fact, the poset is almost ${}^\omega\omega$-bounding.

Definition: Almost ${}^{\omega}\omega$-bounding

The partial order $\mathbb{P}$ is almost ${}^{\omega}\omega$-bounding if

- for every $\mathbb{P}$-name $\dot{f}$ for a function in ${}^{\omega}\omega$ and
- every condition $p \in \mathbb{P}$

there is a ground model function $g \in {}^{\omega}\omega$ such that for every infinite subset of A of ω there is an extension q_A of p such that

$$q_A \Vdash \exists^{\infty} k \in A(\dot{f}(k) \leq \check{g}(k)).$$

Theorem (CH)

The countable support iteration of proper, almost ${}^{\omega}\omega$-bounding posets is weakly bounding.

Theorem (Shelah 1984)

It is relatively consistent that $\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2$.

$\cdots$ which establishes the independence of $\mathfrak{b}$ and $\mathfrak{s}$.

It is worth pointing out that every admissible assignment of $\aleph_1$ and $\aleph_2$ to the cardinal invariants of measure and category in the Cichon diagram can be realized in a generic extension via a countable support iteration of proper posets.

$\cdots$ beyond $\mathfrak{c} = \aleph_2$

The consistency of $\mathfrak{d} = \aleph_1 < \mathfrak{a} = \aleph_2$ is one of the most difficult, persistent problems of the combinatorial cardinal characteristics of the continuum. Note that

- while $\mathfrak{a} < \mathfrak{d}$ holds in the Cohen model,
- the consistency of $\aleph_1 < \mathfrak{d} < \mathfrak{a}$ was obtained only after S. Shelah introduced the method of template iterations.

Moreover the consistency of $\mathfrak{d} = \aleph_1 < \mathfrak{a} = \aleph_2$ is still open.

Theorem (S. Shelah, 1996; published 2000)

(GCH) It is relatively consistent that $\aleph_1 < \mathfrak{d} < \mathfrak{a} = \mathfrak{c}$.

In fact, in Shelah's model of $\mathfrak{d} < \mathfrak{a}$, the following holds:

$$\mathfrak{s} = \aleph_1 < \mathfrak{d} = \mathfrak{b} < \mathfrak{a}.$$

In 2016, introducing a new dimension to Shelah's template construction, the notion of a width of a template, V.F. and D. Mejia generalized the above and in particular obtained the following:

$$\aleph_1 < \mathfrak{s} < \mathfrak{b} = \mathfrak{d} < \mathfrak{a} = \mathfrak{c}.$$

1. It seems that, by 2016, in fact a bit earlier, ccc extensions in which $\mathfrak{c} > \aleph_2$ slowly started slowly gaining some popularity.
2. A question, that should be mentioned at this point is the question if the almost disjointness number can be of countable cofinality.
3. Since $\mathfrak{c}$ cannot be of countable cofinality (König, Hilbert), the question clearly calls for a model of $\aleph_\omega < \mathfrak{c}$.

The question was answered to the positive, by J. Brendle, who, modified Shelah's template iteration construction in such a way, that, in particular, Hechler's poset for adding a mad family of cardinality $\aleph_\omega$, $\mathbb{H}(\aleph_\omega)$, appeared a complete suborder of the modified construction. This, led to the consistency of

$$\aleph_1 < \mathfrak{b} = \mathfrak{d} < \mathfrak{a} = \aleph_\omega < \mathfrak{c}$$

We will briefly return both to $\mathbb{H}(\aleph_\omega)$ and $\aleph_\omega$, as a cardinal invariant value, but before that here are some interesting open questions in probably increasing difficulty:

1. Is it consistent that $\aleph_1 < \mathfrak{s} < \mathfrak{b} = \mathfrak{d} < \mathfrak{a}_e \leq \mathfrak{c}$?
2. Is it consistent that $\aleph_1 < \mathfrak{s} < \mathfrak{b} = \mathfrak{d} < \mathfrak{a}_g \leq \mathfrak{c}$?
3. Is it consistent that $\aleph_1 < \mathfrak{s} < \mathfrak{b} = \mathfrak{d} < \mathfrak{a} = \aleph_\omega < \mathfrak{c}$?
4. Is it consistent that $\aleph_1 < \mathfrak{s} < \mathfrak{b} < \mathfrak{d} < \mathfrak{a}$?

1. V.F. and A. Törnquist developed a combinatorial analogue of $\mathbb{H}(\gamma)$ (Hechler's poset for adding an almost disjoint family of size γ, maximal for $\gamma \geq \omega_1$) for
   - eventually different families of functions,
   - eventually different families of permutations,
   - cofinitary groups.

2. These posets can be used to obtain a modified template construction and produce the consistency of each of (V.F., A. Törnquist, 2015):

$$\mathfrak{a}_e = \aleph_\omega, \mathfrak{a}_p = \aleph_\omega, \mathfrak{a}_g = \aleph_\omega.$$

3. It seems very reasonable to expect a positive answer to the first two questions. It is also reasonable to expect a positive answer to the third question, making the fourth one, the very difficult one!

Question

Is it consistent that $\aleph_1 < \mathfrak{b} < \mathfrak{s}$?

Recall Shelah's creature poset $\mathbb{Q}$.

Definition

Let $\mathscr{C}$ be a family of pure conditions in $\mathbb{Q}$. Then $\mathbb{Q}(\mathscr{C})$ is the suborder of $\mathbb{Q}$ consisting of all $(u, T) \in \mathbb{Q}$ such that

$$\exists R \in \mathscr{C}(R \leq T).$$

Definition

A family of pure conditions $\mathscr{C}$ is centered if whenever $X, Y \in \mathscr{C}$ there is $R \in \mathscr{C}$ which is their common extension.

Theorem (V.F., J. Steprans, 2008)

Let κ be a regular uncountable cardinal, cov$(\mathscr{M}) = \kappa$,

$$\mathscr{H} \subseteq {}^{\omega}\omega$$

is an unbounded, $\leq^*$-directed family of cardinality κ. Assume that

$$\forall \lambda < \kappa (2^{\lambda} \leq \kappa).$$

Then, there is a centered family

$$\mathscr{C}_{\mathscr{H}}$$

of pure conditions, such that $|\mathscr{C}_{\mathscr{H}}| = \kappa$ and such that

1. $\Vdash_{\mathbb{Q}(\mathscr{C}_{\mathscr{H}})}$ "$\mathscr{H}$ is unbounded"
2. $\mathbb{Q}(\mathscr{C}_{\mathscr{H}})$ adds a real not split by the ground model reals.

Theorem (V.F., J. Sterpans, 2008)

(GCH) Let κ be a regular uncountable. Then there is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$.

1. Can we do better? Is it consistent that there is an arbitrarily large spread between $\mathfrak{b}$ and $\mathfrak{s}$?
2. The techniques leading to the above result, heavily use the fact that the cardinality of the independent family is κ!
3. So, we do need a new approach. But, before that, an observation:

Lemma (V.F., B. Irrgang, 2010)

Let $\mathscr{C}$ be a centered family of pure conditions in $\mathbb{Q}$. Then $\mathbb{Q}(\mathscr{C})$ is densely embedded in $\mathbb{M}(\mathscr{F}_{\mathscr{C}})$, where

$$\mathscr{F}_{\mathscr{C}} = \{X \in [\omega]^{\omega} : \exists T \in \mathscr{C}(\mathrm{int}(T) \subseteq X)\}.$$

Proof

The mapping $(u, T) \mapsto (u, \mathrm{int}(T))$ is a dense embedding. □

Corollary

Let κ be a regular uncountable cardinal, $\text{cov}(\mathscr{M}) = \kappa$, $\mathscr{H} \subseteq {}^{\omega}\omega$ is an unbounded, $\leq^*$ directed family of cardinality κ. Assume that $\forall\lambda < \kappa(2^{\lambda} \leq \kappa)$. Then, there is an ultrafilter $\mathscr{U}_{\mathscr{H}}$ such that

$$\Vdash_{\mathbb{M}(\mathscr{U}_{\mathscr{H}})} \text{“}\mathscr{H} \text{ is unbounded”}.$$

Can we do better?

Definition: Hechler's poset for adding a dominating real, $\mathbb{D}$:

The poset consists of all pairs $(s, f) \in {}^{<\omega}\omega \times {}^{\omega}\omega$ such that $(s_1, f_1) \leq (s_2, f_2)$ iff

- s_2 is an initial segment of s_1
- for all $i \in \mathrm{dom}(s_1) \setminus \mathrm{dom}(s_2)$, $s_1(i) \geq f_2(i)$;
- for all $i \in \omega$, $f_2(i) \leq f_1(i)$.

Definition: Hechler's poset for adding a mad family, $\mathbb{H}(\gamma)$:

Let γ be an ordinal. Then, $\mathbb{H}(\gamma)$ is the poset of all finite partial functions

$$p : \gamma \times \omega \to 2$$

such that $\operatorname{dom}(p) = F_p \times n_p$ where $F_p \in [\gamma]^{<\omega}$, $n_p \in \omega$.

The order is given by $q \leq p$ if

1. $p \subseteq q$
2. $|q^{-1}(1) \cap F^p \times \{i\}| \leq 1$ for all $i \in n_q \backslash n_p$.

Strong maximality and elimination of intruders

Definition: Strong Maximality

Let $M \subseteq N$ be models of set theory, $\mathscr{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\omega]^\omega$ and let $A \in N \cap [\omega]^\omega$. Then $(*^{M}_{\mathscr{B}}\,{}^{N}_{A})$ holds if for every

$$h : \omega \times [\gamma]^{<\omega} \to \omega$$

$h \in M$ and every $m \in \omega$ there are $n \geq m$ and $F \in [\gamma]^{<\omega}$ such that

$$[n, h(n,F)) \setminus \bigcup_{\alpha \in F} B_\alpha \subseteq A$$

.

Strong maximality and elimination of intruders

Lemma

Let $G_{\gamma+1}$ be $\mathbb{H}(\gamma+1)$-generic, $G_\gamma = G_{\gamma+1} \cap \mathbb{H}(\gamma)$ and $A_\gamma = \{A_\alpha\}_{\alpha<\gamma}$, where

$$A_\alpha = \{i : \exists p \in G_{\gamma+1} p(\alpha, i) = 1\},$$

for $\alpha \leq \gamma$. Then

$$(*^{V[G_\gamma]}_{\mathscr{A}_\gamma} \ {}^{V[G_{\gamma+1}]}_{A_\gamma})$$

holds.

Lemma: Elimination of Intruders

Let $(*^{M}_{\mathscr{B}} \ {}^{N}_{A})$ hold, $\mathscr{B} = \{B_\alpha\}_{\alpha<\gamma}$, let $\mathscr{I}(\mathscr{B})$ be the ideal generated by $\mathscr{B}$ and the finite sets and let $B \in M \cap [\omega]^\omega, B \notin \mathscr{I}(\mathscr{B})$. Then

$$|A \cap B| = \aleph_0.$$

Assume GCH and let $\kappa < \lambda$ be regular uncountable cardinals. Let

$$f : \{\eta < \lambda : \eta \equiv 1 \mod 2\} \to \kappa$$

be an onto mapping such that

$$\forall \alpha < \kappa, f^{-1}(\alpha) \text{ is cofinal in } \lambda.$$

Recursively define a system of finite support iterations

$$\langle\langle \mathbb{P}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta \leq \lambda\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta < \lambda\rangle\rangle$$

as follows:

- For all α, ζ let $V_{\alpha,\zeta} = V^{\mathbb{P}_{\alpha,\zeta}}$.
- If $\zeta = 0$ then for all $\alpha \leq \kappa$, let $\mathbb{P}_{\alpha,0}$ be Hechler's poset for adding an a.d. family $\mathscr{A}_\alpha = \{A_\beta\}_{\beta<\alpha}$. Note that for $\alpha \geq \omega_1$, $\mathscr{A}_\alpha$ is maximal almost disjoint in $V_{\alpha,0}$.
- If $\zeta = \eta + 1$, $\zeta \equiv 1 \mod 2$, then $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathbb{Q}}_{\alpha,\eta} = \mathbb{M}(\dot{\mathscr{U}}_{\alpha,\eta})$ where $\dot{\mathscr{U}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for an ultrafilter and for all $\alpha < \beta \leq \kappa$,

$$\Vdash_{\mathbb{P}_{\beta,\eta}} \dot{\mathscr{U}}_{\alpha,\eta} \subseteq \dot{\mathscr{U}}_{\beta,\eta}.$$

- If $\zeta = \eta + 1$, $\zeta \equiv 0 \mod 2$, then
  - if $\alpha \leq f(\eta)$, $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for the trivial forcing notion.
  - If $\alpha > f(\eta)$ then $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{D}^{V_{f(\eta),\eta}}$.
- If ζ is a limit, then for all $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is the finite support iteration of $\langle \mathbb{P}_{\alpha,\eta}, \dot{\mathbb{Q}}_{\alpha,\eta} : \eta < \zeta \rangle$.

Furthermore, we guarantee that the construction satisfy the following properties:

1. $\forall \zeta \leq \lambda$ and $\forall \alpha < \beta \leq \kappa$,

$$\mathbb{P}_{\alpha,\zeta} \lessdot \mathbb{P}_{\beta,\zeta}.$$

2. For all $\zeta \leq \lambda$, $\forall \alpha < \kappa$ a strong combinatorial property

$$(*^{V_{\alpha,\zeta}}_{\mathscr{A}_\alpha} \quad ^{V_{\alpha+1,\zeta}}_{A_{\alpha+1}})$$

guaranteeing that A_α eliminates intruders over $V_{\alpha,\zeta}$ holds.

Lemma

The construction satisfies that for all $\alpha < \beta \leq \kappa$, and all $\zeta < \eta \leq \lambda$,

$$\mathbb{P}_{\alpha,\zeta} \lessdot \mathbb{P}_{\beta,\eta}.$$

Lemma

For each $\zeta \leq \lambda$:

1. For every $p \in \mathbb{P}_{\kappa,\zeta}$ there is $\alpha < \kappa$ such that $p \in \mathbb{P}_{\alpha,\zeta}$.
2. For every $\mathbb{P}_{\kappa,\zeta}$-name for a real $\dot{f}$ there is $\alpha < \kappa$ such that $\dot{f}$ is a $\mathbb{P}_{\alpha,\zeta}$-name.

Theorem (V. F., J. Brendle, 2010)

$$V_{\kappa,\lambda} \vDash \mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda.$$

Proof: $\mathfrak{a} \leq \kappa$

- We will show that family $\{A_\alpha\}_{\alpha<\kappa}$ remains maximal in $V_{\kappa,\lambda}$.
- Otherwise $\exists B \in V_{\kappa,\lambda} \cap [\omega]^\omega$ such that

$$\forall \alpha < \kappa |B \cap A_\alpha| < \omega.$$

However there is $\alpha < \kappa$ such that

$$B \in V_{\alpha,\lambda} \cap [\omega]^\omega.$$

- Note that $B \notin \mathscr{I}(\mathscr{A}_\alpha)$. Then the elimination intruders property

$$(* \begin{smallmatrix} V_{\alpha,\lambda} & V_{\alpha+1,\lambda} \\ \mathscr{A}_\alpha & A_{\alpha+1} \end{smallmatrix})$$

holds and so $|B \cap A_{\alpha+1}| = \infty$, which is a contradiction.
- Thus, $\mathfrak{a} \leq \kappa$.

Proof: $\kappa \leq \mathfrak{b}$ and so $\mathfrak{b} = \mathfrak{a} = \kappa$

- Let $B \subseteq V_{\kappa,\lambda} \cap {}^{\omega}\omega$ be of cardinality $< \kappa$. Then there are $\alpha < \kappa$, $\zeta < \lambda$ such that $B \subseteq V_{\alpha,\zeta}$.
- Since $\{\gamma : g(\gamma) = \alpha\}$ is cofinal in λ, there is $\zeta' > \zeta$ such that $f(\zeta') = \alpha$.
- Then $\mathbb{P}_{\alpha+1,\zeta'+1}$ adds a real dominating $V_{\alpha,\zeta'} \cap {}^{\omega}\omega$, and so in particular $V_{\alpha,\zeta} \cap {}^{\omega}\omega$.
- Thus B is not unbounded.
- Therefore in $V_{\kappa,\lambda}$, we have that $\mathfrak{b} \geq \kappa$. However $\mathfrak{b} \leq \mathfrak{a}$ and so, in $V_{\kappa,\lambda}$ we have $\mathfrak{b} = \mathfrak{a} = \kappa$.

Proof: $\mathfrak{s} = \lambda$

To see that in $V_{\kappa,\lambda}$, $\mathfrak{s} = \lambda$, note that if

$$S \subseteq V_{\kappa,\lambda} \cap [\omega]^\omega$$

is of cardinality $< \lambda$, then there is $\zeta < \lambda$ such that

$$\zeta = \eta + 1, \zeta \equiv 1 \mod 2$$

and

$$S \subseteq V_{\kappa,\lambda}.$$

Then $\mathbb{M}(\mathscr{U}_{\kappa,\eta})$ adds a real not split by S and so S is not splitting. □

More Questions:

1. Is it consistent that $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$?
2. Is it consistent that $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$?

In fact many, if not most, admissible constellations of the combinatorial cardinal characteristics with three or more distinct values are open!

Thank you for your attention!